Informal Notes on Lanczos
Chapter V, sects. 1–2, pp. 111ff.

The Passage from Newton to Least Action

For convenience, we divide the demonstration into seven parts, as follows:

1. The foundation: d'Alembert's Principle
Newtonian forces acting on a system of particles, expressed in...
Our ultimate aim: to reformulate (them) as the "Lagrangian" and the Principle of Least Action.

2. Convert to a time-integral
Integrate d'Alembert's principle over time

3. Separate the resulting integral into two parts
(corresponding to d'Alembert's *impressed* and *reactive* (inertial) forces.

4. Integrate the "holonomic" term
Label this term "alpha." It represents the impressed force, derivable from a potential, V.

5. Rearrange the "polygenic" terms
Label them "beta" and "gamma." They represent the inertial reactions, which don't conform to any overall law.

6. Show that the "gamma" term gives us the kinetic energy of the system
Rearrange gamma using the product rule. Recognize the resulting expression as the total kinetic energy T.

7. Arrange the result in terms of the Lagrangian L, the action A, and the Principle of Least Action

Detailed Notes on the above Seven Steps

1. The foundation: d'Alembert's Principle
1a. The basic principle.
D'Alembert's Principle is invoked at the outset by Lanczos (hereinafter, "Lz") as the foundation on which this reasoning will stand. D'Alembert's principle is discussed by Lz in the preceding chapter, sections 4.1 and 4.2. We'll begin with a short review, starting with Newton's law of motion:

$$F = ma$$

D'Alembert makes a small alteration in this:

$$F - ma = 0.$$

Though a small move algebraically, it is the seed of much to come. Lz believes it marks the difference between a natural science of *law* and one of *principle*. Readers of Aristotle will recall that *principle, arche,* is embedded in the very definition of *physis*—and is a world apart from the concept of *nomos.*

Lz is interested in whole systems of bodies, so he writes our principle as a sum:

$$\sum_{i=1}^{n} \bar{F}_i - m_i a_i = 0.$$

1b. Virtual work.

More importantly, Lz introduces the principle of virtual work. Since d'Alembert writes two forces that balance to zero, his principle has, in effect, reduced motion to stasis. *But any static system balances at its lowest energy configuration.*

On a graph of energy vs. displacement, the system will settle at a minimum; and there, we know, the tangent will have zero slope. Any vanishingly small change in force from that position will yield zero change in energy; that is, will do *zero work.* Thus d'Alembert's stasis can be written in terms of the principle of virtual work.

Lz writes a variation in displacement as $\delta\vec{R}$
and that in energy as $\delta\bar{\omega}^e$.

The expression for the variation of energy of a system of forces in equilibrium thus becomes:

$$\delta\bar{\omega}^e = \sum_{i=1}^{n}\left[\left(\vec{F}_i - m_i\vec{a}_i\right)\bullet\delta\vec{R}\right] = 0$$

[that is, net force time displacement, for each particle]

The formal mathematics of variations of this sort is termed *the calculus of variations*—hence the title of Lanczos' book. By way of the theorem we are about to study, this becomes basic to general relativity, quantum mechanics, particle physics, and many other aspects of modern science.

Lz has one more step to make in preparing d'Alembert's equation for the demonstration we are about to undertake. Almost breathtakingly, he multiplies d'Alembert's principle by *time.*

2. Convert to a time-integral

As he explains at the outset of Chapter V, Lz has a very special reason for wanting to do this. The impressed force F is, he says, *holonomic,* which means that it derives from a *law* of force expressed by a function, as readers of Newton know wel!

But the forces of reaction are all over the place, or *polygenic* [having multiple origins], and we have no systematic way to deal with them. Since they vary in unmanageable ways in time, there is only one answer to the problem: *swallow them up in a time-integration.* All the details of the motions will be included in one scalar value for their sum over time. With this aim in view, Lz writes d'Alembert's equation as a time integral; and this is the first line of the derivation on which we are about to embark.

$$\int_{t_1}^{t_2} \bar{\omega}^e dt \equiv \int_{t_1}^{t_2} \sum\left[\vec{F}_i - \frac{d}{dt}m\vec{v}\right]\bullet\delta\vec{R}_i\, dt$$

(Notice that Lz writes $\frac{d}{dt}m\vec{v}$ in place of our $m\vec{a}$. We shall follow him in this.)

We might note that this same move is essential to Maxwell in deriving his equations for the electromagnetic field. Whereas the problem of Lz is that he doesn't know the details of the motions of the individual particles which make up the system, the problem for Maxwell is far worse: he knows that in his system there *are* no particles! Though the electromagnetic "ether" conveys essentially all the energy of life on earth, it *has* no particles—that is, no ponderable, "rest" mass). Maxwell must therefore take this same step in order to begin construction of his theory.

3. Separate the resulting integral into two parts

We separate this integral into two integrals, which for a while will go their own ways. One represents the holonomic impressed force function; the other represents the polygenic, intractable forces of reaction. Note that each of these now includes a time-integration, so that the overall statements have the dimensionality of *action* rather than *energy*. Omitting the summation subscripts, we have:

$$\int\sum\left[\vec{F}-\frac{d}{dt}m\vec{v}\right]\bullet\delta\vec{R}\,dt$$
$$\equiv\underbrace{\int\sum\vec{F}\bullet\delta\vec{R}\,dt}_{\text{Holonomic}}-\underbrace{\int\sum\left(\frac{d}{dt}m\vec{v}\right)\bullet\delta\vec{R}\,dt}_{\text{Polygenic}}$$

The second of these terms involves differentiation of a product. Using the product rule,

$$\frac{d}{dv}(xy)=x\frac{dy}{dv}+y\frac{dx}{dv}$$

we can divide the second term further. Applying it to the polygenic term,

$$\frac{d}{dt}\left(m\vec{v}\bullet\delta\vec{R}\right)=\frac{d}{dt}(m\vec{v})\bullet\delta\vec{R}+(m\vec{v})\bullet\frac{d}{dt}\delta\vec{R}$$

Rearranging, we see that the original polygenic term has split into two parts:

$$-\frac{d}{dt}(m\vec{v})\bullet\delta\vec{R}=-\frac{d}{dt}\left(m\vec{v}\bullet\delta\vec{R}\right)+(m\vec{v})\bullet\frac{d}{dt}\delta\vec{R}$$

The integrand now consists of three terms, which we can usefully label α, β, and γ:

$$(\alpha)\quad\left(\vec{F}\bullet\delta\vec{R}\right)\quad:\quad\text{Holonomic}$$
$$\left.\begin{array}{ll}(\beta) & -\dfrac{d}{dt}m\vec{v}\bullet\delta\vec{R}\\[2ex] (\gamma) & (m\vec{v})\bullet\dfrac{d}{dt}\left(\delta\vec{R}\right)\end{array}\right\}\quad:\quad\text{Polygenic}$$

4. Integrate the "holonomic" term

We now play the easy card: integrating the holonomic term. We have already seen that its holonomic character arises from the fact that it can be obtained by differentiation of a potential term, V:

$$\int_{t_1}^{t_2}\sum\left(\vec{F}\bullet\delta\vec{R}\right)dt=-\delta\int_{t}^{t_2}V\,dt$$

Differentiation of a potential is the vector operation *gradient.* Here the resulting vector has as many components as there are elements in the system.

$$\vec{F}=-\nabla V$$

5. Rearrange the "polygenic" terms (β and γ)

(i) the β term:

$$\int_{t}^{t_2} \frac{d}{dt}\left(\sum m\vec{v} \bullet \delta\vec{R}\right) dt$$

It is a definite integral, in the form

$$\int_{t_1}^{t_2} u\, dt = u_2 - u_1 .$$

The β term then yields the values of the integrand at the beginning and at the end of the motion:

$$\left[\sum m\vec{v} \bullet \delta\vec{R}\right]_{t_1}^{t_2}$$

(ii) the γ term:

Gamma is a bit trickier; we begin with a pair of lemmas. Note first that although the differential dv and the variation δv do not have the same meaning,[1] they can be shown to be functionally interchangeable.

Lemma 1: Just as we may write the differential expression

$$v\, dv = \frac{1}{2} d\left(v^2\right) \quad \text{(because } d\left(v^2\right) = 2v dv\text{)}$$

so we have the same relationship in terms of the variation:

$$v\, \delta v = \frac{1}{2} \delta\left(v^2\right)$$

Lemma 2: Again, in the expression appearing in our integrand,

$$\frac{d}{dt} \delta\vec{R}$$

we may write (somewhat surprisingly):

$$\delta\left(\frac{d\vec{R}}{dt}\right) = \delta\vec{v} \qquad \text{(taking } \vec{v} = \frac{d\vec{R}}{dt}\text{)}$$

We can assemble these relationships in the flow diagram on the next page, in which *kinetic energy* emerges from our variational reasoning.

6. Show that the "gamma" term gives us the kinetic energy of the system

Using the lemmas, we transform the gamma term as follows:

[1] The difference between dv and δv may be seen thus: $dv = f(x + dx) - f(x)$, while $\delta v = F(x) - f(x) = \varepsilon\varphi(x)$. Also note commutativity [?]

$$\int \sum m\vec{v} \bullet \frac{d}{dt}\left(\delta\vec{R}\right) dt$$
$$= \int \sum m\vec{v} \bullet \delta \frac{d\vec{R}}{dt}\, dt$$
$$= \int \sum m\vec{v} \bullet \delta\vec{v}\, dt \qquad \text{(Lemma 1)}$$
$$= \int \sum \tfrac{1}{2} m\delta\left(v^2\right) dt \qquad \text{(Lemma 2)}$$
$$= \delta \tfrac{1}{2} \int \sum mv^2 dt.$$

7. Arrange the result in terms of the Lagrangian *L*, the action *A*, and the Principle of Least Action

(i) Recapitulating the results so far,

$$\int_{t_1}^{t_2} \delta\omega^e\, dt = \delta \int_{t_1}^{t_2} \sum_i m_i v_i^2\, dt - \int_{t_1}^{t_2} V\, dt - \left[\sum_i m_i \vec{v}_i \bullet \delta\vec{R}\right]_{t_1}^{t_2}$$

(ii) Define the kinetic energy T.

Define T as the kinetic energy of a moving mass: $\frac{1}{2}mv^2$. Then the kinetic energy term in the above expression becomes:

$$\delta \int_{t_1}^{t_2} T\, dt$$

and the above expression becomes

$$\int_{t_1}^{t_2} \delta\omega^e\, dt = \delta \int_{t_1}^{t_2} (T - V)\, dt - \delta\left[\sum_i m_i \vec{v}_i \bullet \delta\vec{R}_i\right]_{t_1}^{t_2}$$

(iii) Set the boundary values to zero:

The last term in our expression sets the initial and final values of the variation of the system's momentum. We now require that these values be fixed, and hence not subject to variation. In other words, *we must set in advance the final state—the goal—of the system!*

$$\begin{cases} \delta\left[\sum_i m_i \vec{v}_i \bullet \delta\vec{R}_i\right]_{t_1} = 0 \\ \delta\left[\sum_i m_i \vec{v}_i \bullet \delta\vec{R}_i\right]_{t_2} = 0 \end{cases}$$

(iv) Introduce the Lagrangian:

Our expression for the virtual work of the system has been greatly simplified:

$$\int_{t_1}^{t_2} \delta\omega^e\, dt = \delta \int_{t_1}^{t_2} (T - V)\, dt$$

We now simplify further by introducing a new quantity, L, the difference between the kinetic and potential energies of the system:

$$L \equiv T - V$$

so that

$$\int_{t_1}^{t_2} \delta\omega^e \, dt = \delta \int_{t_1}^{t_2} L \, dt$$

(v) Introduce "Action," and remember d'Alembert:
Define the quantity *action* as

$$A = \int L \, dt$$

Remember that our whole effort has been to develop a new form for the expression of d'Alembert's principle of equilibrium—asserted by setting the virtual work equal to zero. We now have an elegant way of asserting d'Alembert's insight:

$$\int \delta\omega^e = \delta \int L \, dt = 0$$

This is the Principle of Least Action, which we may state even more compactly as

$$\delta A = 0$$

Thomas K. Simpson
September 2012